ECE484 Principles of Safe Autonomy
Stability Verification

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Overview

Roadmap

1. Lyapunov Stability
   - Motivation and Examples
   - Stability Notions
   - Ranking Functions and Intuition
   - Lyapunov Direct Method
   - Invariance and Nonlinear Examples
   - Quadratic Lyapunov and LMIs

Roadmap

What Does Stability Mean?

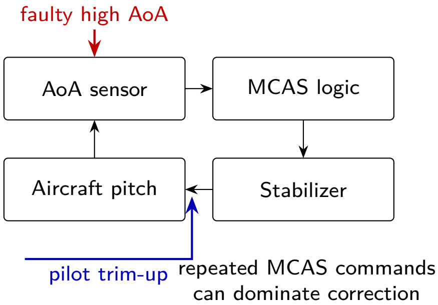
- Key requirement for any control system: stability of the closed-loop equilibrium
- For an equilibrium $x^* = 0$ of $\dot{x} = f(x)$:
  - Lyapunov stable: small perturbations stay small forever
  - Asymptotically stable: Lyapunov stable *and* trajectories converge to 0
  - Unstable: state grows without bound
- Eigenvalue test works for LTI; need a general method for nonlinear systems
- Lyapunov's direct method: use an energy-like function to certify stability

Why Verify Stability? 737 MAX MCAS Anecdote

- Boeing 737 MAX accidents (Lion Air, October 29, 2018; Ethiopian, March 10, 2019).
- A faulty high angle-of-attack signal could trigger MCAS nose-down stabilizer trim.
- Pilots trimmed nose-up, but MCAS could re-activate and command nose-down again.
- Closed-loop effect: repeated divergence from safe pitch

  $$e_{k+1} = e_k + \Delta_{\mathrm{MCAS}} - \Delta_{\mathrm{pilot}},$$

  where Δ_{MCAS} can dominate pilot recovery.
- Takeaway: even with a capable pilot, biased feedback can drive the system outside a recoverable envelope.

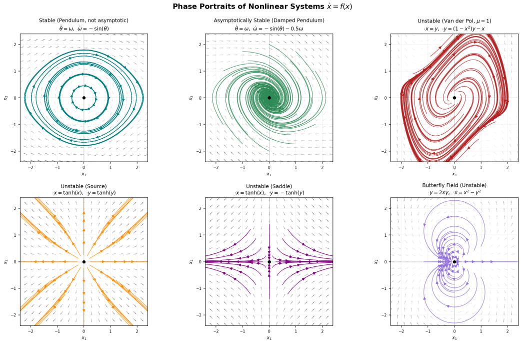


Nonlinear 2D Examples for Lyapunov Analysis

Examples used in this lecture:

- $\dot{\theta} = \omega, \;\; \dot{\omega} = -\sin(\theta)$ (undamped pendulum)
  $\Rightarrow$ stable (Lyapunov), not asymptotically stable.
- $\dot{\theta} = \omega, \;\; \dot{\omega} = -\sin(\theta) - 0.5\omega$ (damped pendulum)
  $\Rightarrow$ origin is asymptotically stable.
- $\dot{x} = y, \;\; \dot{y} = (1 - x^2)y - x$ (Van der Pol, $\mu = 1$)
  $\Rightarrow$ origin is unstable (attracting limit cycle).
- $\dot{x} = \tanh(x), \;\; \dot{y} = \tanh(y)$
  $\Rightarrow$ unstable (source).
- $\dot{x} = \tanh(x), \;\; \dot{y} = -\tanh(y)$
  $\Rightarrow$ unstable (saddle type).
- $\dot{y} = 2xy, \;\; \dot{x} = x^2 - y^2$ (butterfly field)
  $\Rightarrow$ unstable at the origin.

Van der Pol and butterfly are polynomial (local Lipschitz).

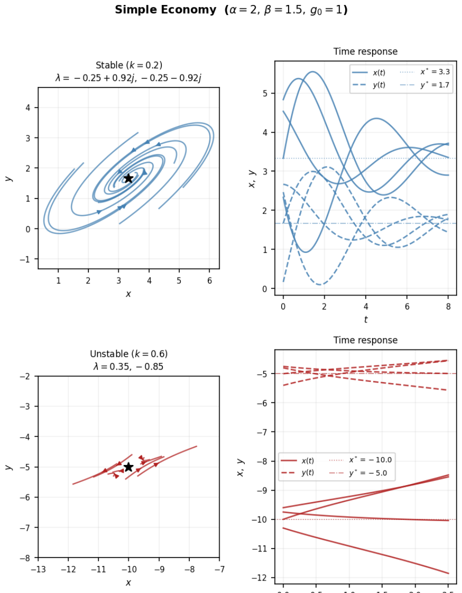


Example: A Simple Economy

Model $(\alpha, \beta, k > 0)$

$$\dot{x} = x - \alpha y$$
$$\dot{y} = \beta(x - y - g)$$

- x: national income
- y: consumer spending rate
- g: government expenditure rate
- Feedback policy: $g = g_0 + kx$
- Does the economy settle to a steady state, or grow without bound?



Economy: Equilibrium and Stability

Find the equilibrium (set $\dot{x} = \dot{y} = 0$, $g = g_0 + kx$):

$$0 = x^* - \alpha y^* \qquad \Rightarrow \quad x^* = \alpha y^*$$
$$0 = \beta((1-k)x^* - y^* - g_0)$$

Substitute $x^* = \alpha y^*$:

$$y^*(\alpha(1-k) - 1) = g_0$$

$$\boxed{x^* = \frac{g_0 \alpha}{\alpha - 1 - k\alpha}, \quad y^* = \frac{g_0}{\alpha - 1 - k\alpha}}$$

Closed-loop matrix (substitute $g = g_0 + kx$):

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} + \text{const}$$

- $\mathrm{tr}(A) = 1 - \beta$; $\det(A) = \beta(\alpha(1-k) - 1)$
- Stable iff $\beta > 1$ and $k < 1 - \frac{1}{\alpha}$
- Gain k shifts eigenvalues — policy design!

Roadmap

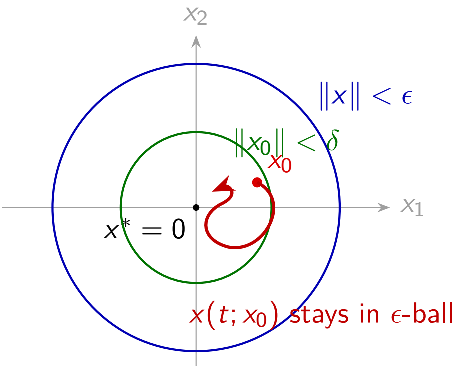
Lyapunov Stability: Definition

Definition (Lyapunov stable)

The equilibrium $x^* = 0$ of $\dot{x} = f(x)$ is Lyapunov stable if

$$\forall\, \epsilon > 0, \quad \exists\, \delta > 0 \quad \text{s.t.} \quad \|x(0)\| < \delta \;\Rightarrow\; \|x(t; x_0)\| < \epsilon \quad \forall\, t \geq 0.$$

- The trajectory stays inside a ball of radius ϵ for all time



Asymptotic and Global Asymptotic Stability

Definition (Asymptotically stable)

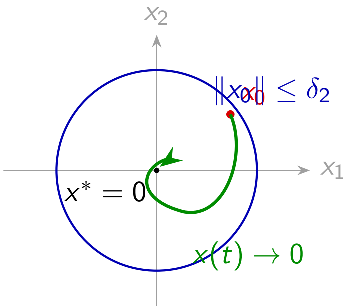
$x^* = 0$ is asymptotically stable if it is Lyapunov stable and

$$\exists\, \delta_2 > 0 \quad \text{s.t.} \quad \forall x_0,\ \|x_0\| \le \delta_2 \Rightarrow \lim_{t\to\infty} x(t; x_0) = 0.$$

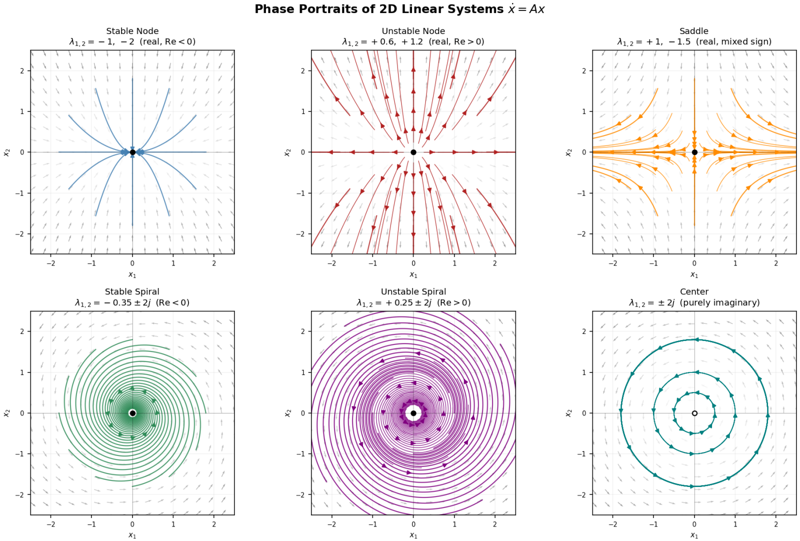
Definition (Globally asymptotically stable)

$x^* = 0$ is globally asymptotically stable if it is Lyapunov stable and

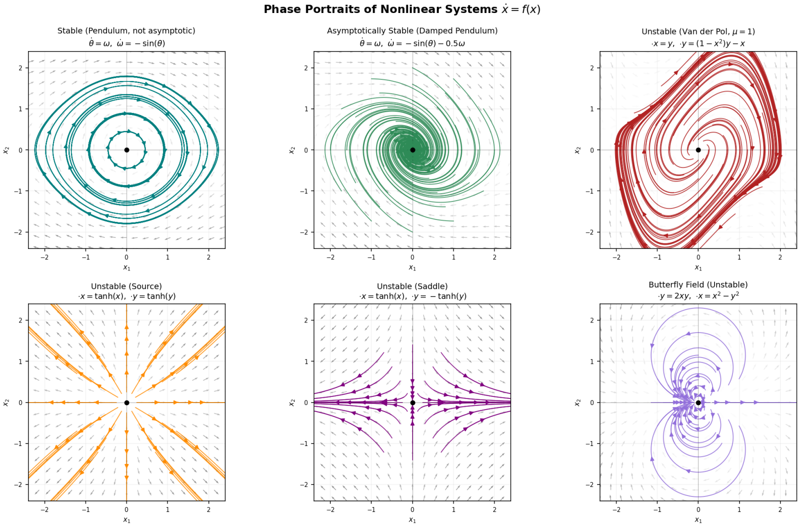
$$\forall x_0 \in \mathbb{R}^n, \quad \lim_{t\to\infty} x(t; x_0) = 0.$$



Lyapunov and Asymptotically Stable Linear Systems



Lyapunov and Asymptotically Stable Nonlinear Systems



Roadmap

Program Termination: 2D Example

- **Question:** how can we prove a program always terminates?

Example (one component can increase)

$$\texttt{while}\ (x > 0)\ \texttt{or}\ (y > 0)\ \texttt{do}\quad \begin{cases} x := x - 1,\ y := y + 2, & x > 0, \\ y := y - 1, & x = 0,\ y > 0. \end{cases}$$

$$\text{State } (x, y) \in \mathbb{N}^2_{\geq 0}, \qquad T = \{(0, 0)\}.$$

- In general: undecidable (Halting Problem) to decide termination for all programs.

Theorem (Ranking-function sufficient condition)

Consider a deterministic transition system $x_{k+1} = F(x_k)$ on state set $\mathcal{S}$ with terminal set $T \subseteq \mathcal{S}$. If there exists $V : \mathcal{S} \to \mathbb{N}$ such that

$$x \notin T \Rightarrow V(F(x)) < V(x),$$

then every execution reaches T in finite time (the program terminates).

Ranking Function Proof and Discrete Phase Portrait

Choose $V(x,y) = 3x + y$.

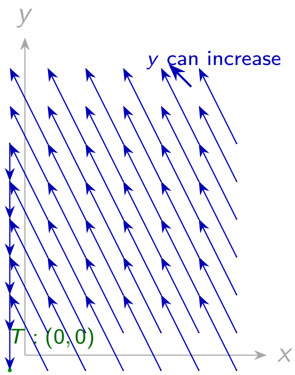
- If $x > 0$: $(x', y') = (x-1, y+2)$, so

$$V(x',y') = 3(x-1) + (y+2) = V(x,y) - 1.$$

- If $x = 0,\ y > 0$: $(x', y') = (x, y-1)$, so

$$V(x',y') = y - 1 = V(x,y) - 1.$$

$$(x,y) \notin T \Rightarrow V(F(x,y)) < V(x,y),$$
$$V(x,y) \in \mathbb{N}.$$



- Strict integer descent cannot continue forever.
- Therefore the loop reaches $T = \{(0,0)\}$ in finite steps.

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Lyapunov's Direct Method: Candidate and Derivative

- **Parallel with ranking functions:**
  - Discrete proof of termination: $V(F(x)) < V(x)$ (strict descent in $\mathbb{N}$)
  - Continuous analog for stability: choose $V(x)$ with $\dot{V}(x) < 0$
- Find a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with:
  - $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$
  - V is continuously differentiable
- For $\dot{x} = f(x)$ and trajectory $x(t; x_0)$, define:

$$\begin{aligned}\dot{V}(x(t;x_0)) &:= \frac{d}{dt} V(x(t;x_0)) \\ &= \frac{\partial V}{\partial x}(x(t;x_0))\,\frac{d\,x(t;x_0)}{dt} \\ &= \nabla V(x(t;x_0))^\top f(x(t;x_0)).\end{aligned}$$

Lyapunov Theorem

Theorem (Lyapunov Direct Method)

Consider

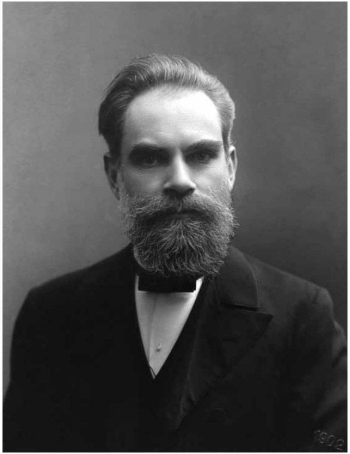
$$\dot{x} = f(x), \qquad f(0) = 0.$$

Suppose there exists $V \in C^1$ such that $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$.

- If $\dot{V}(x) \leq 0$ for all $x \in D \setminus \{0\}$, then $x^* = 0$ is Lyapunov stable.
- If $\dot{V}(x) < 0$ for all $x \in D \setminus \{0\}$, then $x^* = 0$ is asymptotically stable.
- If the above hold on all $\mathbb{R}^n$ and V is radially unbounded

$$V(x) \to \infty \text{ as } \|x\| \to \infty,$$

then $x^* = 0$ is globally asymptotically stable.



Aleksandr Lyapunov (1857–1918)

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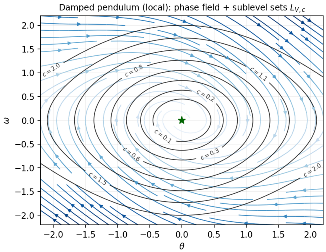
Lyapunov Functions and Invariants

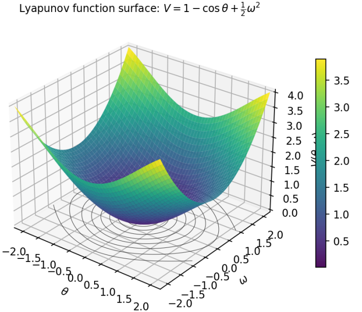
- Sublevel sets of V:
  $L_{V,c} := \{x \in \mathbb{R}^n \mid V(x) \leq c\}$.
- A set $\mathcal{I}$ is invariant if
  $x(0) \in \mathcal{I} \Rightarrow x(t) \in \mathcal{I},\ \forall t \geq 0$.
- If $\dot{V}(x) \leq 0$ and $x(0) \in L_{V,c}$, then

$$V(x(t)) \leq V(x(0)) \leq c \ \Rightarrow\ x(t) \in L_{V,c},\ \forall t \geq 0.$$

- Therefore, $\mathcal{I} := L_{V,c}$ is invariant.
- More generally, if an initial set $\mathcal{X}_0 \subseteq L_{V,c}$, then $Reach(\mathcal{X}_0) \subseteq L_{V,c}$.



Local phase field + sublevel sets $L_{V,c}$.



Example: Undamped Pendulum

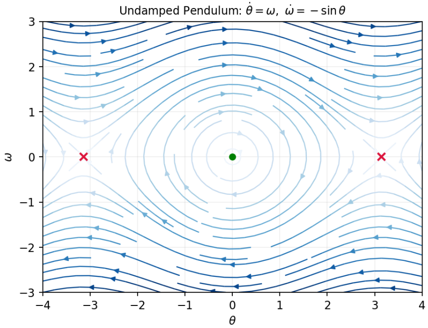
- System:

$$\begin{bmatrix} \dot\theta \\ \dot\omega \end{bmatrix} = \begin{bmatrix} \omega \\ -\sin\theta \end{bmatrix}$$

- Candidate (energy): $V(\theta, \omega) = 1 - \cos\theta + \frac{1}{2}\omega^2$
  - Not radially unbounded in θ
  - Not globally positive definite, local conclusions.
- Derivative:

$$\begin{aligned} \dot V(\theta, \omega) &= \nabla V(\theta, \omega)^\top f(\theta, \omega) \\ &= \begin{bmatrix} \sin\theta \\ \omega \end{bmatrix}^\top \begin{bmatrix} \omega \\ -\sin\theta \end{bmatrix} = (\sin\theta)\omega + \omega(-\sin\theta) \\ &= 0. \end{aligned}$$

- Around $(0,0)$ (e.g. $|\theta| < \pi$): V is positive definite and $\dot V = 0 \Rightarrow$ Lyapunov stable, not asymptotically stable.
- Not global to $(0,0)$: there are other equilibria at $\theta = 2k\pi, \omega = 0$ and $\theta = (2k+1)\pi, \omega = 0$.



Green dots: centers at $\theta = 2k\pi$. Red crosses: saddles at $\theta = (2k+1)\pi$.

Example: Damped Pendulum

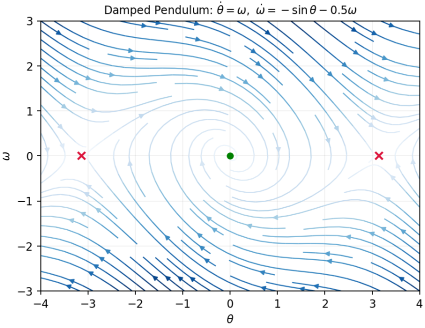
- System:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\sin\theta - 0.5\,\omega \end{bmatrix}$$

- Use the same candidate: $V(\theta, \omega) = 1 - \cos\theta + \frac{1}{2}\omega^2$
- Same observations
- Derivative (same $\nabla V^\top f$ style):

$$\begin{aligned} \dot{V}(\theta, \omega) &= \nabla V(\theta, \omega)^\top f(\theta, \omega) \\ &= \begin{bmatrix} \sin\theta \\ \omega \end{bmatrix}^\top \begin{bmatrix} \omega \\ -\sin\theta - 0.5\,\omega \end{bmatrix} \\ &= (\sin\theta)\omega + \omega(-\sin\theta - 0.5\,\omega) \\ &= -0.5\,\omega^2 \leq 0. \end{aligned}$$

- In a neighborhood of $(0,0)$ (e.g. $|\theta| < \pi$): this gives local stability and (with invariance arguments) local asymptotic stability.
- Not global to $(0,0)$: other equilibria exist at $\theta = 2k\pi,\ \omega = 0$, $k \neq 0$.



Green dots: stable equilibria at $\theta = 2k\pi$. Red crosses: saddles at $\theta = (2k+1)\pi$.

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Positive Definite Matrices

- Definition: A symmetric matrix P is positive definite, $P \succ 0$, $x^\top P x > 0, \forall x \neq 0$
- Equivalently, all eigenvalues of P are positive.
- Key property: $P \succ 0 \Rightarrow x^\top P x$ is a sum of squares (SOS).
- Norm bound: $\lambda_{\min}(P)\|x\|^2 \leq x^\top P x \leq \lambda_{\max}(P)\|x\|^2$.
- Example 1: $P = I \succ 0$,

$$V(x) = x^\top I x = x^\top x = x_1^2 + x_2^2 \quad \text{(SOS)}.$$

- Example 2:

$$P = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix} \succ 0, \qquad V(x) = x^\top P x = 2x_1^2 + x_1 x_2 + x_2^2 = 2\left(x_1 + \frac{x_2}{4}\right)^2 + \frac{7}{8}x_2^2.$$

LTI Lyapunov Theorem

Theorem (Lyapunov equation)

For $\dot{x} = Ax$, A is Hurwitz iff $\forall Q \succ 0$, $\exists\ P \succ 0$ s.t. $A^\top P + PA = -Q$ and $V(x) = x^\top Px$ is a Lyapunov function for the system.

Proof (Forward direction).

Assume there exist $P \succ 0$, $Q \succ 0$ with $A^\top P + PA = -Q$. Define $V(x) = x^\top Px$. Then $V(x) > 0$ for all $x \neq 0$, and

$$\dot{V}(x) = \frac{d}{dt}(x^\top Px) = \dot{x}^\top Px + x^\top P\dot{x} = x^\top(A^\top P + PA)x = -x^\top Qx < 0, \quad \forall x \neq 0.$$

Hence V is positive definite and strictly decreases along nonzero trajectories, so the origin is globally asymptotically stable. Therefore A is Hurwitz. □

Reverse direction omitted here.

Simple Economy via Lyapunov (Alternative to Eigenvalues)

- Shift equilibrium to origin:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \dot{x} = Ax, \quad A = \begin{bmatrix} 1 & -2 \\ 1.2 & -1.5 \end{bmatrix}.$$

- Linear Matrix Inequality (LMI) in P:

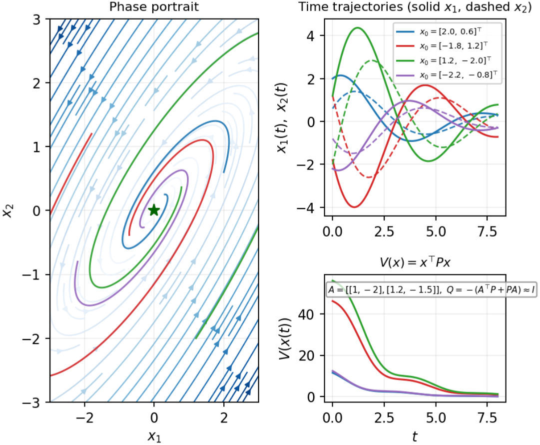
$$A^\top P + PA \preceq -I, \qquad P = P^\top \succ 0.$$

- Lyapunov LMIs are convex and polynomial-time; SDP solvers scale to low-hundreds of states; beyond 1K+ dimensional systems need to exploit structure.
- For this example, fix $Q = I$ and solve the equality

$$A^\top P + PA = -Q, \qquad P = P^\top \succ 0.$$

- Solution:

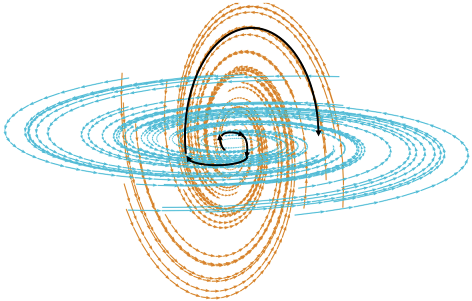
$$P = \begin{bmatrix} 5.1 & -4.6667 \\ -4.6667 & 6.5556 \end{bmatrix} \succ 0,$$



Phase portrait + trajectories, with x_1, x_2, and V vs time.

Further Reading

- Lyapunov Theory for Hybrid/Switched Systems: switching between individually stable modes may still violate safety/stability without common Lyapunov functions or dwell-time conditions. *Pointers: Liberzon, Switching in Systems and Control (Birkhäuser, 2003); Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems" (IEEE TAC, 1998).*
- Neural Lyapunov Functions: learn candidate certificates with neural networks, then verify $\dot{V} < 0$ (or SOS/SMT relaxations) to obtain certified regions. *Pointers: Chang, Roohi, & Gao, "Neural Lyapunov Control" (NeurIPS, 2019).*
- Lyapunov for Vision-Based Control: stability with perception in the loop, e.g., image-feature dynamics and learned visual-servoing policies. *Pointers: Chaumette & Hutchinson, "Visual servo control, Part I: Basic approaches" (IEEE RAM, 2006); Chaumette & Hutchinson, "Visual servo control, Part II: Advanced approaches" (IEEE RAM, 2007).*



Switching can destabilize trajectories without extra guarantees.

Summary and Control-Design Takeaways

- Lyapunov analysis turns stability into a certificate: $V > 0$ and $\dot{V} \leq 0$ (stable), $\dot{V} < 0$ (asymptotically stable).
- Sublevel sets $L_{V,c}$ induce invariant sets $\mathcal{I}$, which define safe operating envelopes.
- In autonomy stacks, invariants support runtime monitoring, supervisory switching, and fail-safe fallback logic.
- Always separate local vs global guarantees and align controller deployment with the certified region of attraction.
- For LTI models, quadratic $V(x) = x^\top P x$ with $A^\top P + PA \preceq -Q$ gives a convex LMI-based design/test workflow.
- Practical pipeline: model (or linearize), design controller, certify with V, then enforce constraints in planning/control.